

FINITELY PRESENTABLE SUBGROUPS AND ALGORITHMS

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ABSTRACT. We show that there is a finitely presented group G for which there is no algorithm that takes a finitely presentable subgroup H of G , an abstract finite presentation Q for H , and constructs an isomorphism between Q and H . To do this, we show that there is a finitely presented group with solvable word problem (namely, the Baumslag-Solitar group $BS(2,3)$) for which there is no algorithm that takes a recursive presentation of that same group, and solves the word problem in the recursive presentation. Our main result suggests, but does not prove, that there is a genuine difference between strong effective coherence and weak effective coherence.

1. INTRODUCTION

Decision problems for finitely presented groups are well-studied, going back to the work of Max Dehn [7] in 1911. It is of some surprise that a finite data set which completely describes a group can give rise to so many uncomputable problems. The word problem, of deciding when a word represents the trivial element in a group, was the first of these to be well understood, with the independent work of Boone, Britton and Novikov [3, 4, 15]. A clever adaption of this problem by the independent work of Adian and Rabin [1, 16] gives rise to the impossibility of the isomorphism problem, of deciding if two finite presentations describe isomorphic groups. To list all major results in group-theoretic decision problems would be a substantial undertaking, and we refer the reader to [14] for an excellent survey of the discipline up to 1992.

What we concern ourselves with in this paper are algorithmic problems relating to subgroups of finitely presented groups. The membership problem, of deciding if a word lies in the subgroup generated by a finite set of elements, is known to be unsolvable for certain finitely presented groups, even in cases when the word problem for the group is solvable, as shown by Miller [13, Theorem 21]. Our interest lies in computing presentations of subgroups, and computing maps between subgroups and abstract presentations of them.

We begin by investigating the problem of determining finite presentations of finitely presentable subgroups. We show this is algorithmically impossible in general. If P is a group presentation, then \overline{P} denotes the group it presents.

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Corollary 3.4. *There is a finite presentation $P = \langle X|R \rangle$ of a group for which there is no algorithm that, on input of a finite collection of words $w_1, \dots, w_n \in X^*$ such that $\langle w_1, \dots, w_n \rangle^{\overline{P}}$ (the subgroup of \overline{P} generated by $\{w_1, \dots, w_n\}$) is finitely presentable, outputs a finite presentation $\langle Y|S \rangle$ defining a group isomorphic to $\langle w_1, \dots, w_n \rangle^{\overline{P}}$.*

We note that the above group \overline{P} can even be taken to have solvable word problem (see the remark after corollary 3.4 in section 3). This result motivates us to study the relationship between finite and infinite presentations of the same group. As it turns out there can be some very unexpected behaviour, mostly due to non-Hopfian groups; a Hopfian group is one with no proper quotient isomorphic to itself.

Theorem 3.12. *There exists a finite presentation $P = \langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$, and a recursive presentation $T = \langle x_1, \dots, x_n | r_1, r_2, \dots \rangle$ (where r_i in P are the same as r_i in T for all $i \leq m$) such that:*

1. $\overline{P} \cong \overline{T}$, and hence \overline{T} is finitely presentable.
2. The surjective homomorphism $\overline{\phi} : \overline{P} \rightarrow \overline{T}$ induced from the map $x_i \mapsto x_i$ is not an isomorphism.

It is this strange behaviour that eventually allows us to prove our main result: that it is algorithmically impossible to construct an explicit isomorphism between a finite set of words in a group, and an abstract finite presentation which is known to be isomorphic to the subgroup generated by those words. Along the way, we establish that a solution to the word problem for a finite presentation cannot be uniformly lifted to a solution to the word problem for a recursive presentation of the same group. We note however that such a solution can be uniformly lifted if both presentations are finite.

Theorem 4.1. *There is a finite presentation P of a group with solvable word problem (namely, the Baumslag-Solitar group $BS(2,3)$) for which there is no algorithm that, on input of a recursive presentation Q such that $\overline{P} \cong \overline{Q}$, outputs a solution to the word problem for Q .*

Of course, having an explicit isomorphism between P and Q would give us a solution to the word problem for \overline{Q} . This leads us to the following result.

Theorem 4.2. *There is a finite presentation $P = \langle X|R \rangle$ of a group with solvable word problem (namely, $BS(2,3)$) for which there is no algorithm that, on input of a recursive presentation $Q = \langle Y|S \rangle$ such that $\overline{P} \cong \overline{Q}$, outputs a set map $\phi : X \rightarrow Y^*$ which extends to an isomorphism $\overline{\phi} : \overline{P} \rightarrow \overline{Q}$.*

By viewing isomorphisms between groups as Tietze transformations rather than set maps (see [12, §1.5] for a full exposition of this concept), we can interpret theorem 4.2 in the following way.

Theorem 4.4. *There is a finite presentation P of a group with solvable word problem (namely, $\text{BS}(2,3)$) for which there is no algorithm that, on input of a recursive presentation Q such that $\overline{P} \cong \overline{Q}$, outputs a recursive enumeration of Tietze transformations of type (T1) and (T2) (manipulation of relators), and a finite set of Tietze transformations of type (T3) and (T4) (addition and removal of generators), with notation as per [12, §1.5], transforming P to Q .*

A direct consequence of the Higman embedding theorem [10, Theorem 1] is that any recursively presented group can be uniformly embedded into a finitely presented group. By a careful application of this result, paying special attention to the uniformity in which such a finite presentation is constructed as given in the proof of [18, Theorem 12.18], we show the main result of this paper.

Theorem 4.6. *There is a finitely presented group G (with finite presentation $P = \langle X|R \rangle$) for which there is no algorithm that, on input of a finite collection of words $\{w_1, \dots, w_n\} \subset X^*$ such that $\langle w_1, \dots, w_n \rangle^{\overline{P}}$ is finitely presentable, and a finite presentation $Q = \langle Y|S \rangle$ such that $\overline{Q} \cong \langle w_1, \dots, w_n \rangle^{\overline{P}}$, outputs a set map $\phi: Y \rightarrow \{w_1, \dots, w_n\}^*$ which extends to an isomorphism $\overline{\phi}: \overline{Q} \rightarrow \langle w_1, \dots, w_n \rangle^{\overline{P}}$ (that is, the homomorphism $\overline{\phi}: \overline{Q} \rightarrow \overline{P}$ is injective, and $\text{Im}(\overline{\phi}) = \langle w_1, \dots, w_n \rangle^{\overline{P}}$).*

However, like most incomputability results in group theory, the above is a somewhat contrived example. For many cases, there is such an algorithm as described above. The most immediate, and broadest, of these is the class of locally Hopfian groups; groups whose finitely generated subgroups are all Hopfian. So we have a very strong positive analogue to the above result.

Theorem 3.11. *In the class of locally Hopfian groups, the algorithm from theorem 4.6 does indeed exist, uniformly over the entire class. Hence, in this class of groups, weak effective coherence is equivalent to strong effective coherence (see definitions 3.1 and 3.2 in section 3).*

The above two results address a typographical error in Groves and Wilton [8, Definition 1.1], where they define effective coherence as weak effective coherence (definition 3.1 in this work), instead of strong effective coherence (definition 3.2 in this work) which they later corrected it to in [9, Definition 1.2]. One could naively assume that the notions of weak effective coherence and strong effective coherence are equivalent; this work actually came from a reading of [8] and a realisation that the definitions may indeed be different.

We note that all results in [8] were proved for the case of strongly coherent groups, and in light of the typographical correction provided in [9] we neither suggest nor imply that there are any errors with the main conclusions of [8]. However, if one takes the definition of effective coherence in [8] to be weak effective coherence, then the proofs of [8, Lemma 1.2] and subsequent remark [8, Remark 1.3] contain a subtle argument that, in light of theorem 4.6 in this work, cannot be justified.

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2. PRELIMINARIES

2.1. Notation. With the convention that \mathbb{N} contains 0, we take φ_m to be the m^{th} partial recursive function $\varphi_m : \mathbb{N} \rightarrow \mathbb{N}$, and the m^{th} partial recursive set (r.e. set) W_m as the domain of φ_m . If $P = \langle X|R \rangle$ is a group presentation with generating set X and relators R , then we denote by \overline{P} the group presented by P . A presentation $P = \langle X|R \rangle$ is said to be a *recursive presentation* if X is a finite set and R is a recursive enumeration of relators; P is said to be an *infinite recursive presentation* if instead X is a recursive enumeration of generators. A group G is said to be *finitely presentable* if $G \cong \overline{P}$ for some finite presentation P . If P, Q are group presentations then we denote their free product presentation by $P * Q$, which is given by taking the disjoint union of their generators and relators; this extends to the free product of arbitrary collections of presentations. If X is a set, then we denote by X^{-1} a set of the same cardinality as X (considered an ‘inverse’ set to X) along with a fixed bijection $\phi : X \rightarrow X^{-1}$, where we denote $x^{-1} := \phi(x)$. We write X^* for the set of finite words on $X \cup X^{-1}$, including the empty word \emptyset . If g_1, \dots, g_n are a collection of elements of a group G , then we write $\langle g_1, \dots, g_n \rangle^G$ for the subgroup in G generated by these elements, and $\langle\langle g_1, \dots, g_n \rangle\rangle^G$ for the normal closure of these elements in G . Given an algebraic property ρ of groups, a group G is said to be *locally ρ* if every finitely generated subgroup of G is ρ . A group G is said to be *Hopfian* if any surjective homomorphism $f : G \twoheadrightarrow G$ is necessarily injective, and *coherent* if every finitely generated subgroup is finitely presentable (i.e., G is locally finitely presentable). Finally, the commutator $[x, y]$ is taken to be $xyx^{-1}y^{-1}$.

2.2. Recursion theory. Recalling *Cantor’s pairing function* $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, $\langle x, y \rangle := \frac{1}{2}(x+y)(x+y+1)+y$ which is a computable bijection from $\mathbb{N} \times \mathbb{N}$ to \mathbb{N} , one can extend this inductively to define a bijection from \mathbb{N}^n to \mathbb{N} by $\langle x_1, \dots, x_n \rangle := \langle \langle x_1, \dots, x_{n-1} \rangle, x_n \rangle$. We note that this function, and all its extensions, are recursively computable. The following two results can be found as [17, §1.8 Theorem V] and [17, §11.2 Theorem I] respectively.

Theorem 2.1 (The s-m-n theorem [17, §1.8 Theorem V]). *For all $m, n \in \mathbb{N}$, a partial function $f : \mathbb{N}^{m+n} \rightarrow \mathbb{N}$ is partial-recursive if and only if there is a recursive function $s : \mathbb{N}^m \rightarrow \mathbb{N}$ such that, for all $e_1, \dots, e_m, x_1, \dots, x_n \in \mathbb{N}$ we have that $f(e_1, \dots, e_m, x_1, \dots, x_n) = \varphi_{s(e_1, \dots, e_m)}(\langle x_1, \dots, x_n \rangle)$.*

Theorem 2.2 (Kleene recursion theorem [17, §11.2 Theorem I]). *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a recursive function. Then there exists $n \in \mathbb{N}$ with $\varphi_n = \varphi_{f(n)}$.*

We use these to make the following observation in recursion theory, which is essential to the proof of our main result.

Lemma 2.3. *There is no partial recursive function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that, given n satisfying $|W_n| < \infty$, we have that $g(n)$ halts with $g(n) = |W_n|$. That is, given a recursively enumerable set W_n which is finite, we can't, in general, recursively find the size of W_n .*

This particular result is quite interesting, as the parallel question in group theory: ‘given a finite presentation P of a finite group, can we find $|\overline{P}|$?’ is algorithmically decidable, as discussed in [14].

Proof. Assume such a g exists. Define $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ by

$$f(n, m) := \begin{cases} 1 & \text{if } g(n) \text{ halts and } m \leq g(n) \\ \uparrow & \text{in all other cases} \end{cases}$$

Then f is partial-recursive, since g is. By theorem 2.1, there exists a recursive function $s : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(n, m) = \varphi_{s(n)}(m)$ for all m, n . Since s is recursive, theorem 2.2 shows that there must be some n' such that $\varphi_{s(n')} = \varphi_{n'}$. Thus $f(n', m) = \varphi_{n'}(m)$ for all $m \in \mathbb{N}$. Moreover, $\varphi_{n'}(m)$ can only halt on the cases $m \leq g(n')$ (if at all), and no others. So $|W_{n'}|$ is finite. By definition of g , we thus have that $g(n') \downarrow = |W_{n'}|$. Hence by construction, $W_{n'} = \{0, \dots, g(n')\}$ and so $|W_{n'}| = g(n') + 1$, a contradiction since $g(n') = |W_{n'}|$. \square

We finish by showing that given an index n with W_n finite, we can recursively compress it to $\{0, \dots, |W_n| - 1\}$

Lemma 2.4. *There is a recursive function $h : \mathbb{N} \rightarrow \mathbb{N}$ satisfying the following:*

1. *If $W_n = \emptyset$ then $W_{h(n)} = \emptyset$.*
2. *If $1 \leq |W_n| < \infty$ then $W_{h(n)} = \{0, \dots, |W_n| - 1\}$.*
3. *If $|W_n| = \infty$ then $W_{h(n)} = \mathbb{N}$.*

Proof. Given n , we begin an enumeration of W_n . For each element enumerated into W_n we increase the size of $W_{h(n)}$ by 1, by adding the next smallest number not already in $W_{h(n)}$. If $W_n = \emptyset$ then $W_{h(n)} = \emptyset$. If $1 \leq |W_n| < \infty$, then $W_{h(n)} = \{0, \dots, |W_n| - 1\}$. If $|W_n| = \infty$ then we will continue to sequentially enumerate elements of \mathbb{N} into $W_{h(n)}$, so $W_{h(n)} = \mathbb{N}$. As this is an effective description of $W_{h(n)}$, we have that h is recursive. \square

2.3. Groups. We begin with a known result about the word problem for HNN extensions of groups.

Lemma 2.5. *Let $H, K \leq G$ be isomorphic finitely generated subgroups of G , each having solvable membership problem in G . Let $\varphi : H \rightarrow K$ be an isomorphism. Then the HNN extension $G *_\varphi$ has solvable word problem.*

Proof. This is immediate from the normal form theorem for HNN extensions (see [11, §IV Theorem 2.1]). \square

The following group construction from [2] plays an important part in our arguments, as it is our main example of a finitely presented non-Hopfian group.

Definition 2.6. The *Baumslag-Solitar groups* $\overline{\text{BS}}(m, n)$ are defined via the finite presentations

$$\text{BS}(m, n) := \langle s, t \mid s^{-1}t^m s = t^n \rangle$$

which are each an HNN extension of \mathbb{Z} .

Theorem 2.7 (Baumslag-Solitar [2, Theorem 1]). *The group $\overline{\text{BS}}(2, 3)$ is non-Hopfian and has solvable word problem. More specifically, the map $\overline{f} : \overline{\text{BS}}(2, 3) \rightarrow \overline{\text{BS}}(2, 3)$ given by extending the map $f : \{s, t\} \rightarrow \{s, t\}^*$, $f(s) = s$, $f(t) = t^2$, is a non-injective epimorphism. The word $[s^{-1}ts, t]$ is non-trivial in $\overline{\text{BS}}(m, n)$ and lies in the kernel of \overline{f} .*

Proof. That $\overline{\text{BS}}(2, 3)$ has solvable word problem comes from the fact that it is an HNN extension of \mathbb{Z} (see lemma 2.5, or alternatively a result by Magnus in [11, § IV Theorem 5.3] that all 1-relator groups have solvable word problem). The remainder of the theorem is proved in [2, Theorem 1]. \square

Definition 2.8. Given a word $w(s, t) \in \{s, t\}^*$, we define the set map $g : \{s, t\}^* \rightarrow \{s, t\}^*$ by

$$g(w(s, t)) := \begin{cases} [s^{-1}ts, t] & \text{if } w = \emptyset \\ w(s, [s^{-1}, t]) & \text{otherwise (i.e., replace } s \mapsto s, t \mapsto [s^{-1}, t]) \end{cases}$$

Then denote by w_i the word $g^i(\emptyset)$ (that is, g applied i times to \emptyset), where $w_0 := \emptyset$.

Corollary 2.9. *For any $i \in \mathbb{N}$, the extension $\overline{f}^i : \overline{\text{BS}}(2, 3) \rightarrow \overline{\text{BS}}(2, 3)$ of the i^{th} iteration f^i of the map f given in theorem 2.7 is also a non-injective epimorphism. Moreover, $w_j \in \ker(\overline{f}^i)$ if and only if $j \leq i$.*

Proof. A finite iteration of a non-injective epimorphism is again a non-injective epimorphism. For the second part, note that $f([s^{-1}, t]) = t$ in $\overline{\text{BS}}(2, 3)$, so $f(w_j) = w_{j-1}$ in $\overline{\text{BS}}(2, 3)$ whenever $j \geq 1$. So, since $w_1 = [s^{-1}ts, t]$ which is non-trivial in $\overline{\text{BS}}(2, 3)$ but lies in $\ker(\overline{f})$, we have our desired result. \square

2.4. Enumerable processes in groups. We describe some partial algorithms and recursively enumerable sets, in the context of group presentations. These are somewhat straightforward observations, but we point them out for the convenience of any reader not completely familiar with the area.

Lemma 2.10. *Let $P = \langle X \mid R \rangle$ be a recursive presentation. Then the words in X^* which represent the identity in \overline{P} are recursively enumerable. Moreover, this algorithm is uniform over all recursive presentations.*

Proof. A word $w \in X^*$ represent the identity in \overline{P} if and only if w takes the form $\prod_{i=1}^n w_i r_{j_i}^{\epsilon_i} w_i^{-1}$ for some $n \in \mathbb{N}$, some $w_i \in X^*$, some $r_{j_i} \in R \cup \{\emptyset\}$ and some $\epsilon_i \in \{-1, 1\}$. Now since all such words of this form are recursively enumerable by simply running through all possible combinations of n, w_i, r_{j_i} and ϵ_i , we have our desired result. \square

Lemma 2.11. *There is a partial algorithm that, on input of two finite presentations $P = \langle X \mid R \rangle$ and $Q = \langle Y \mid S \rangle$, and a set map $\phi : X \rightarrow Y^*$, halts if and only if ϕ extends to a homomorphism $\overline{\phi} : \overline{P} \rightarrow \overline{Q}$.*

Proof. For each $r \in R$, use lemma 2.10 to check if $\overline{\phi}(r) = e$ in \overline{Q} . This process will halt if and only if all of these do, thus if and only if $\overline{\phi}$ is a homomorphism. \square

Lemma 2.12. *There is a partial algorithm that, on input of two finite presentations $P = \langle X|R \rangle$ and $Q = \langle Y|S \rangle$, halts if and only if $\overline{P} \cong \overline{Q}$, and outputs an isomorphism between them.*

Proof. Begin an enumeration of all set maps $\phi_i : X \rightarrow Y^*$, and similarly an enumeration of all set maps $\psi_j : Y \rightarrow X^*$. Now in parallel begin checking which of the ϕ_i 's and ψ_j 's extend to homomorphisms with lemma 2.11. For each such pair, use lemma 2.10 to begin checking if $\overline{\phi_i} \circ \overline{\psi_j}(x) = x$ in \overline{P} for all $x \in X$, and if $\overline{\psi_j} \circ \overline{\phi_i}(y) = y$ in \overline{Q} for all $y \in Y$. Such an overall process will eventually halt if and only if $\overline{P} \cong \overline{Q}$; if it does halt then the maps ϕ_i and ψ_j that it halts on give us our isomorphisms. \square

It is important to note that the proof of lemma 2.12 does not hold if we instead consider recursive presentations. In fact, even if we start with one recursive presentation, and one finite presentation, the lemma does not hold; we show this later as theorem 4.2.

Corollary 2.13. *Let $P = \langle X|R \rangle$ be a finite presentation. Then the set of finite presentations defining groups isomorphic to \overline{P} is recursively enumerable. Moreover, this enumeration algorithm is uniform over all finite presentations.*

Proof. Begin an enumeration of all finite presentations P_1, P_2, \dots . For each such presentation P_i , use lemma 2.12 to test if $\overline{P_i} \cong \overline{P}$; for all (and only) such presentations satisfying this will the process will eventually halt. \square

3. INITIAL QUESTIONS AND OBSERVATIONS

3.1. Impossibility of determining presentations. We begin by exploring the following problems of algorithmically determining information about presentations of subgroups. This was motivated by considering [8], which defines a finitely generated group G to be *effectively coherent* if it is coherent (all of its finitely generated subgroups are finitely presentable) and there is an algorithm for G that, on input of a finite collection of words S , outputs a finite presentation for the subgroup generated by S . This was corrected in [9], matching our definition 3.2 below. From this we are led to make the following two definitions:

Definition 3.1. We say a coherent group $G = \overline{\langle X|R \rangle}$ is *weakly effectively coherent* if there is an algorithm for G that, on input of a finite collection of words $S \subseteq X^*$, outputs a finite presentation for the subgroup $\langle S \rangle^G$ generated by S .

Definition 3.2. We say a coherent group $G = \overline{\langle X|R \rangle}$ is *strongly effectively coherent* if there is an algorithm for G that, on input of a finite collection of words $S \subseteq X^*$, outputs a finite presentation $P = \langle Y|V \rangle$ for the subgroup $\langle S \rangle^G$ generated by S , along with a map $\phi : Y \rightarrow X^*$ which extends to an injection $\overline{\phi} : \overline{P} \rightarrow G$ whose image is $\langle S \rangle^G$.

It is immediate that all strongly effectively coherent groups are also weakly effectively coherent; we ask the question of whether every group is weakly effectively coherent? The following two results quickly resolve this question.

Proposition 3.3. *There is a finite presentation $P = \langle X|R \rangle$ of a group for which there is no algorithm that, on input of a finite collection of words $w_1, \dots, w_n \in X^*$ such that $\langle w_1, \dots, w_n \rangle^{\overline{P}}$ is finitely presentable, decides if $\langle w_1, \dots, w_n \rangle^{\overline{P}}$ is perfect or not.*

Proof. Take $P = \langle X|R \rangle$ to be a finite presentation of a group with unsolvable word problem (see [18, Lemma 12.7]), and suppose such an algorithm exists for this presentation. Note that for any word $w \in X^*$, we have that $\langle w \rangle^{\overline{P}}$ is cyclic, and hence finitely presentable. So, on input of a single word w , the algorithm will decide if $\langle w \rangle^{\overline{P}}$ is perfect or not. But a cyclic group is perfect if and only if it is trivial. Hence the algorithm will decide if $w = e$ in \overline{P} or not, which is impossible since \overline{P} has unsolvable word problem. \square

Corollary 3.4. *There is a finite presentation $P = \langle X|R \rangle$ of a group for which there is no algorithm that, on input of a finite collection of words $w_1, \dots, w_n \in X^*$ such that $\langle w_1, \dots, w_n \rangle^{\overline{P}}$ is finitely presentable, outputs a finite presentation $\langle Y|S \rangle$ defining a group isomorphic to $\langle w_1, \dots, w_n \rangle^{\overline{P}}$.*

Proof. We need only note that there is a uniform algorithm which, on input of a finite presentation of a group, decides if that presentation defines a perfect group or not, by forming the abelianisation and testing if that is trivial or not (see [14, p. 10]). This, combined with the hypothesis, would contradict proposition 3.3. \square

Remark. As pointed out to the author by Chuck Miller, the work of Collins in [6] shows that the group in corollary 3.4 can be taken to have solvable word problem. This is because such an algorithm as described in corollary 3.4 would explicitly solve the order problem, of deciding what the order of an element of the group is. However, a direct consequence of [6, Theorem A] is that there is a finitely presented group with solvable word problem and unsolvable order problem.

So now it now seems natural to ask the following question:

Question 1. Are strong effective coherence and weak effective coherence equivalent notions for finitely presented groups?

Eventually we will see that many intuitive ways to attempt to show equivalence actually fail.

3.2. Identifying a subgroup with an abstract presentation of it. Having shown it is impossible to determine presentations of subgroups, we shift our attention to the question of finding subgroups. By considering the trivial group, and the fact that the triviality problem is undecidable, one can show that there is no algorithm that, given two finite presentations P, Q , determines if \overline{P} embeds in \overline{Q} or not. The following two stronger results were obtained in [5].

Theorem 3.5 (Chiodo [5, Theorem 6.8]). *There is a finitely presented group G such that the set of finite presentations of groups which embed into G is not recursively enumerable.*

Theorem 3.6 (Chiodo [5, Theorem 6.6]). *There is no algorithm that, on input of two finite presentations $P = \langle X|R \rangle$, $Q = \langle Y|S \rangle$ such that \overline{P} embeds in \overline{Q} , outputs an explicit map $\phi: X \rightarrow Y^*$ which extends to an embedding $\overline{\phi}: \overline{P} \hookrightarrow \overline{Q}$.*

In the proof of theorem 3.6, as found in [5], we see that the algorithmic problem arises from not definitely knowing a set of target words for an injection from \overline{P} into \overline{Q} . But suppose we somehow knew which elements \overline{P} maps on to in \overline{Q} . Is this enough to now construct the embedding? With this in mind, and recalling the definitions of a weakly (resp. strongly) effectively coherent group, we are motivated to ask the following question.

Question 2. For each finite presentation $P = \langle X|R \rangle$ of a group is there an algorithm that, on input of a finite collection of words $\{w_1, \dots, w_n\}$ such that $\langle w_1, \dots, w_n \rangle^{\overline{P}}$ is finitely presentable, and a finite presentation $Q = \langle Y|S \rangle$ such that $\overline{Q} \cong \langle w_1, \dots, w_n \rangle^{\overline{P}}$, outputs a set map $\phi: Y \rightarrow \{w_1, \dots, w_n\}^*$ which extends to an isomorphism $\overline{\phi}: \overline{Q} \rightarrow \langle w_1, \dots, w_n \rangle^{\overline{P}}$. That is, the homomorphism $\overline{\phi}: \overline{Q} \rightarrow \overline{P}$ is injective, and $\text{Im}(\overline{\phi}) = \langle w_1, \dots, w_n \rangle^{\overline{P}}$.

In other words, for each finite presentation P , if we are given a set of words w_1, \dots, w_n which generate a finitely presentable subgroup of \overline{P} , and an abstract finite presentation Q with $\overline{Q} \cong \langle w_1, \dots, w_n \rangle^{\overline{P}}$, can we construct an isomorphism from \overline{Q} to $\langle w_1, \dots, w_n \rangle^{\overline{P}}$, or vice-versa? We note this has the following immediate consequence:

Proposition 3.7. *If G is a group for which there exists an algorithm as described in question 2, then G is strongly effectively coherent if and only if it is weakly effectively coherent.*

Proof. Comes immediately from the definitions of weak/strong effective coherence (definitions 3.1 and 3.2). \square

Question 2 was originally considered after the author read the proof of [8, Lemma 1.2] and subsequent remark [8, Remark 1.3] where, were it not for a later correction of a typographical error as provided in [9], it would suggest that given an abstract finite presentation of a finitely presentable subgroup implies one can algorithmically construct an isomorphism between the two (that is, implying that the answer to question 2 is yes, moreover, uniformly over all finitely presented groups). This would, in turn, imply that weak and strong effective coherence are equivalent notions.

For the groups eventually considered in [8] there was a straightforward algorithm, seeing as all subgroups were Hopfian (which we show possible in corollary 3.11 below). In addition, for all subsequent results shown in [8] all weakly effectively coherent groups are actually proven to be strongly effectively coherent, so combined with the typographical correction from [9] this then becomes a moot point. However, [8, Lemma 1.2] and subsequent remark [8, Remark 1.3] do raise an interesting question as to how close strong effective coherence and weak effective coherence actually are.

What follows is a useful observation, which shows that the ‘obvious’ method one would try to use to resolve question 2 in the positive does not lead to a

solution. We note that it is a *very* subtle point that this is not a solution (so subtle in fact that we initially thought it was a solution). However, we first need the following standard result about finitely presentable groups.

Lemma 3.8. *Let $\langle X|R \rangle$ be a recursive presentation of a finitely presentable group. Then there is a finite subset $R' \subseteq R$ such that $\overline{\langle X|R \rangle} \cong \overline{\langle X|R' \rangle}$ via extending the identity map on X . That is, there is a finite truncation R' of R such that all other relations are a consequence of the first R' .*

We use this to show the following.

Proposition 3.9. *Let $P = \langle X|R \rangle$ be a finite presentation of a group, $\{w_1, \dots, w_n\} \subset X^*$ a finite collection of words such that $\langle w_1, \dots, w_n \rangle^{\overline{P}}$ is finitely presentable, and $Q = \langle Y|S \rangle$ a finite presentation such that $\overline{Q} \cong \langle w_1, \dots, w_n \rangle^{\overline{P}}$. Then there is an algorithm to output a finite set of words $\{c_1, \dots, c_k\} \subset \{w_1, \dots, w_n\}^*$ such that each c_i is trivial in \overline{P} and $\langle w_1, \dots, w_n | c_1, \dots, c_k \rangle$ is isomorphic to \overline{Q} (and hence $\langle w_1, \dots, w_n \rangle^{\overline{P}}$).*

Proof. Begin an enumeration c_1, c_2, \dots of all words in the w_i 's which are trivial in \overline{P} (this is straightforward: begin a list of all words in the w_i 's as words in the x_i 's, as well as a list of all words in the x_i 's which are trivial in \overline{P} . This can be done by repeated application of lemma 2.10. Whenever a word appears in both lists, include it). Define the finite presentation $P_i := \langle w_1, \dots, w_n | c_1, \dots, c_i \rangle$. By lemma 3.8, as $\langle w_1, \dots, w_n \rangle^{\overline{P}}$ is finitely presentable, and $T := \langle w_1, \dots, w_n | c_1, c_2, \dots \rangle$ is a recursive presentation of a group isomorphic to $\langle w_1, \dots, w_n \rangle^{\overline{P}}$ via extension of the map $w_i \mapsto w_i$, there exists some finite m such that $\overline{P}_m \cong \overline{T}$ (again via extension of the map $w_i \mapsto w_i$). That is, using lemma 3.8 we can truncate the relations of T at position m (forming P_m) such that all successive relations are consequences of the first m (note that selecting m is not an algorithmic process; for the moment we merely appeal to the fact that such an m exists). So we have $\overline{Q} \cong \langle w_1, \dots, w_n \rangle^{\overline{P}} \cong \overline{T} \cong \overline{P}_m$, where Q is our given explicit finite presentation. Now we use lemma 2.12 to begin checking for an isomorphism between \overline{Q} and $\overline{P}_1, \overline{P}_2, \dots$. Eventually this process will stop at some k (perhaps different from m) such that $\overline{P}_k \cong \overline{T} \cong \overline{Q}$. \square

Note that we were very careful to mention that the selection of m in the above proof was existential, but not necessarily recursive. This is very important, as later in corollary 4.3 we construct a class of presentations for which the selection of such an m is provably non-recursive.

Remark. With the above proof in mind, one would naively hope that the map from $P_k = \langle w_1, \dots, w_n | c_1, \dots, c_k \rangle$ to $P = \langle X|R \rangle$ given by extending $w_i \mapsto w_i \in X^*$ would be an injection onto its image $\langle w_1, \dots, w_n \rangle^{\overline{P}}$; in the case that \overline{P}_k is Hopfian this would indeed be true, as corollary 3.11 below shows. This would resolve question 2. However we shall eventually show that this need not be the case in general, by constructing a counterexample with the use of finitely presented non-Hopfian groups.

Theorem 3.10. *Let P , P_k and $\{w_1, \dots, w_n\}$ be as in the proof of proposition 3.9. If $\langle w_1, \dots, w_n \rangle^{\overline{P}}$ is Hopfian, then the map $\phi : \{w_1, \dots, w_n\} \rightarrow X^*$ sending w_i (as a generator of P_k) to w_i (as a word in X^*) extends to a monomorphism $\overline{\phi} : \overline{P}_k \hookrightarrow \overline{P}$ (and thus to an isomorphism to its image).*

Proof. By the proof of proposition 3.9, we know that ϕ extends to a surjection $\overline{\phi} : \overline{P}_k \rightarrow \langle w_1, \dots, w_n \rangle^{\overline{P}}$. But $\overline{P}_k \cong \langle w_1, \dots, w_n \rangle^{\overline{P}}$, which is Hopfian. Hence $\overline{\phi}$ must be injective, and thus an isomorphism. \square

Corollary 3.11. *In the class of locally Hopfian groups, the answer to question 2 is yes (uniformly over the entire class). Hence, in this class of groups, weak effective coherence is equivalent to strong effective coherence.*

Proof. Simply combine proposition 3.7 with theorem 3.10. \square

We now note an important observation of this paper.

Proposition 3.12. *There exist finite presentations $P = \langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$ and $T = \langle x_1, \dots, x_n | r_1, \dots, r_m, s_1, \dots, s_k \rangle$ (where r_i in P are the same as r_i in T for all $i \leq m$) such that:*

1. $\overline{P} \cong \overline{T}$.
2. *The quotient map $\overline{\phi} : \overline{P} \rightarrow \overline{T}$ induced from the map $x_i \mapsto x_i$ is not an isomorphism.*

Observe that this quotient map is always an isomorphism in the case of Hopfian groups, by very definition.

Proof. Take $P = \text{BS}(2, 3) = \langle s, t | s^{-1}t^2s = t^3 \rangle$ to be a finite presentation of the Baumslag-Solitar group from definition 2.6. Recall the map f and word $w := [s^{-1}ts, t]$ both from theorem 2.7. Then $e \neq w \in \ker(\overline{f})$, so the recursive presentation $Q := \langle s, t | s^{-1}t^2s = t^3, w, \ker(\overline{f}) \rangle$ is such that $\overline{Q} \cong \overline{P} / \ker \overline{f} \cong \overline{P}$ from theorem 2.7. So \overline{Q} is finitely presentable, thus by lemma 3.8 there is a finite collection of words $s_2, \dots, s_k \in \ker(\overline{f})$ such that $\overline{Q} \cong \langle s, t | s^{-1}t^2s = t^3, w, s_2, \dots, s_k \rangle$ via the map $s \mapsto s, t \mapsto t$; set $T := \langle s, t | s^{-1}t^2s = t^3, w, s_2, \dots, s_k \rangle$. Then we have that $\overline{P} \cong \overline{T}$, but not via the extension of the map $s \mapsto s, t \mapsto t$, as $w = e$ in \overline{T} but $w \neq e$ in \overline{P} . \square

Corollary 3.13. *There exists a finite presentation $P = \langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$ and a recursive presentation $T = \langle x_1, \dots, x_n | r_1, r_2, \dots \rangle$ (where r_i in P are the same as r_i in T for all $i \leq m$) such that:*

1. $\overline{P} \cong \overline{T}$, and hence \overline{T} is finitely presentable.
2. *The surjective homomorphism $\overline{\phi} : \overline{P} \rightarrow \overline{T}$ induced from the map $x_i \mapsto x_i$ is not an isomorphism.*

The astute reader will have noticed that all we have done in the above corollary is re-engineer the notion of non-Hopfian groups in a seemingly convoluted manner. The point of doing so is to stress the following:

1. There may be many ways to truncate the relators of T to get a presentation of a group isomorphic to \overline{P} .
2. Truncating T to get a finite presentation T' with $\overline{T'} \cong \overline{T}$ may not give a presentation which is isomorphic to \overline{P} via the quotient map we have been discussing.

4. MAIN RESULTS

With the aid of the non-Hopfian group $\overline{\text{BS}}(2, 3)$, combined with our observations in recursion theory, we can now resolve question 2. We begin by showing that a solution to the word problem can't be algorithmically lifted from a finite presentation of $\overline{\text{BS}}(2, 3)$ to a recursive presentation of $\overline{\text{BS}}(2, 3)$.

Theorem 4.1. *There is a finite presentation P of a group with solvable word problem (namely, $\text{BS}(2, 3)$) for which there is no algorithm that, on input of a recursive presentation Q such that $\overline{P} \cong \overline{Q}$, outputs a solution to the word problem for Q .*

Proof. Assume we have such an algorithm. We proceed to contradict lemma 2.3. With the map f from theorem 2.7, we can form a recursive enumeration of $\ker(\overline{f^i})$ (that is, all words in $\{s, t\}^*$ which lie in the kernel of $\overline{f^i}$). Note that $\ker(\overline{f^i}) \not\subseteq \ker(\overline{f^j})$ whenever $i < j$, as \overline{f} is a non-injective epimorphism. So, given n with $|W_n| < \infty$, we consider the r.e. set $W_{h(n)}$ from lemma 2.4 and define the recursive presentation $P_n := \langle s, t \mid s^{-1}t^2s = t^3, \ker(\overline{f^{i+1}}) \ \forall i \in W_{h(n)} \rangle$. Note that $w_1(s, t)$ (from definition 2.8) is trivial in \overline{P}_n if and only if $W_n \neq \emptyset$. Also, $\overline{\text{BS}}(2, 3) \cong \overline{P}_n$ since $|W_n| < \infty$ (we have just taken the homomorphic image of $|W_n|$ iterations of \overline{f}). Moreover, as the $\ker(\overline{f^i})$ are nested and $W_{h(n)} = \{0, \dots, |W_n| - 1\}$, we have that $P_n = \langle s, t \mid s^{-1}t^2s = t^3, \ker(\overline{f^{|W_n|}}) \rangle$ (where we set $\overline{f^0}$ to be the identity map). Since we have a solution to the word problem for $\overline{\text{BS}}(2, 3)$ (by theorem 2.7), we can use our assumed algorithm to construct a solution to the word problem for \overline{P}_n , uniformly in n . Now test to see if $w_0(s, t), w_1(s, t), \dots$ are trivial in \overline{P}_n , in order. Let k be the largest number such that $w_k(s, t) = e$ in \overline{P}_n , then $|W_n| = k$ (by corollary 2.9). So we have an algorithm that, on input of n with $|W_n| < \infty$, outputs $|W_n|$. But this is impossible, by lemma 2.3. \square

Since preparing this paper, we were shown by Chuck Miller a more direct proof of theorem 4.1, without the use of as many preceding lemmas in group theory and recursion theory. We include our proof concurrently with Chuck's (the latter being essentially an extension of our proposition 3.12), as we believe that the techniques we have introduced are interesting in their own right.

Proof of theorem 4.1 (Chuck Miller). Assume we have such an algorithm. Take $P = \langle X \mid R \rangle$ to be a finite presentation for $\overline{\text{BS}}(2, 3)$. Fix $Q = \langle X \mid R \cup S \rangle$ to be a finite presentation of a non-trivial quotient which is isomorphic to \overline{P} , but not by the quotient map induced by $\text{Id}_X : X \rightarrow X$ (say, instead, by the extension of the map $\phi : X \rightarrow X^*$). Fix a word $w \in X^*$ such that w lies in the kernel of $\overline{\phi}$, but is not trivial in $\overline{\text{BS}}(2, 3)$. Given any r.e. set W_i , we form the recursive presentation $P_{i,j} := \langle X \mid R \cup S \text{ if } j \in W_i \rangle$. That is, $P_{i,j} := \langle X \mid R \rangle$ if $j \notin W_i$, and $P_{i,j} := \langle X \mid R \cup S \rangle$ if $j \in W_i$. So $P_{i,j}$ is a recursive presentation (we add all the relators S to Q if we see $j \in W_i$). Now use our assumed algorithm that solves the word problem in $P_{i,j}$ to test if $\overline{\phi}(w) = e$ in $\overline{P}_{i,j}$; this will occur if and only if $j \in W_i$. By taking W_i to be non-recursive, we derive a contradiction. \square

Theorem 4.2. *There is a finite presentation $P = \langle X|R \rangle$ of a group with solvable word problem (namely, $\text{BS}(2,3)$) for which there is no algorithm that, on input of a recursive presentation $Q = \langle Y|S \rangle$ such that $\overline{P} \cong \overline{Q}$, outputs a set map $\phi : X \rightarrow Y^*$ which extends to an isomorphism $\overline{\phi} : \overline{P} \rightarrow \overline{Q}$.*

Proof. Suppose such an algorithm exists. Given a recursive presentation $Q = \langle Y|S \rangle$ such that $\overline{P} \cong \overline{Q}$, use our supposed algorithm to output a set map $\phi : X \rightarrow Y^*$ which extends to an isomorphism $\overline{\phi} : \overline{P} \rightarrow \overline{Q}$. But since we have a solution to the word problem for P , we can combine this with the map ϕ to get a solution to the word problem for Q , thus contradicting theorem 4.1. \square

At this point it seems relevant to mention that, by lemma 3.8, there will always be a finite subset S' of S such that all other relators are consequences of S' , and hence $\overline{\langle Y|S \rangle} \cong \overline{\langle Y|S' \rangle}$. So we have the following immediate corollary:

Corollary 4.3. *There is a finite presentation $P = \langle X|R \rangle$ of a group with solvable word problem (namely, $\text{BS}(2,3)$) for which there is no algorithm that, on input of a recursive presentation $Q = \langle Y|S \rangle$ such that $\overline{P} \cong \overline{Q}$, outputs a finite subset $S' \subseteq S$ such that all other relators $S \setminus S'$ are consequences of S' . Equivalently, construction of a finite truncation point m of S (with m as in the proof of proposition 3.9) is not algorithmically possible in general.*

As an aside, if we were instead to view isomorphisms between groups as Tietze transformations rather than set maps (see [12, §1.5] for a full exposition of this concept) then we can interpret theorem 4.2 in the following way.

Theorem 4.4. *There is a finite presentation P of a group with solvable word problem (namely, $\text{BS}(2,3)$) for which there is no algorithm that, on input of a recursive presentation Q such that $\overline{P} \cong \overline{Q}$, outputs a recursive enumeration of Tietze transformations of type (T1) and (T2) (manipulation of relators), and a finite set of Tietze transformations of type (T3) and (T4) (addition and removal of generators), with notation as per [12, §1.5], transforming P to Q .*

Remark. It should be pointed out that such a sequence of Tietze transformations as described above will always exist, which follows directly from lemma 3.8. We merely truncate the relators of Q to get a finite presentation Q' , perform a finite sequence of Tietze transformations which takes P to Q' (possible by [12, Corollary 1.5]), and then add rest of the enumeration of relators of Q to Q' . The point here is that we cannot compute an enumeration of such a sequence in general.

Before proceeding to our main result, we note the following lemma which shows that the direction in which we define our isomorphism in theorem 4.2 is inconsequential.

Lemma 4.5. *Using the notation of theorem 4.2, having an explicit set map $\phi : X \rightarrow Y^*$ which extends to an isomorphism $\overline{\phi}$ allows us to construct a set map $\psi : Y \rightarrow X^*$ which extends to the inverse of $\overline{\phi}$ (and hence is an isomorphism). The reverse also holds.*

Proof. This follows from the fact that we only deal with recursive presentations, and we can uniformly enumerate all trivial words of such presentations by lemma 2.10. \square

We now have all the technical machinery required to prove our main result in this work: that we cannot uniformly construct an isomorphism between a finitely presentable subgroup of a finitely presented group, and an abstract finite presentation of that subgroup.

Theorem 4.6. *There is a finitely presented group G (with finite presentation $P = \langle X|R \rangle$) for which there is no algorithm that, on input of a finite collection of words $\{w_1, \dots, w_n\} \subset X^*$ such that $\langle w_1, \dots, w_n \rangle^{\overline{P}}$ is finitely presentable, and a finite presentation $Q = \langle Y|S \rangle$ such that $\overline{Q} \cong \langle w_1, \dots, w_n \rangle^{\overline{P}}$, outputs a set map $\phi : Y \rightarrow \{w_1, \dots, w_n\}^*$ which extends to an isomorphism $\overline{\phi} : \overline{Q} \rightarrow \langle w_1, \dots, w_n \rangle^{\overline{P}}$ (that is, the homomorphism $\overline{\phi} : \overline{Q} \rightarrow \overline{P}$ is injective, and $\text{Im}(\overline{\phi}) = \langle w_1, \dots, w_n \rangle^{\overline{P}}$).*

Note that this is actually possible uniformly over all finite presentations of groups whenever the subgroup in question is Hopfian, as noted in corollary 3.11.

Proof. Suppose we had such an algorithm; we proceed to contradict theorem 4.2. Take a recursive enumeration $P_i := \langle X_i|R_i \rangle$ of all recursive presentations of groups (that is, each X_i is a finite set, and each R_i is a recursive enumeration of words in X_i^*). By the Higman embedding theorem (see [18, Theorem 12.18]) we can embed their free product (with presentation $P_1 * P_2 * \dots$) into a finitely presented group with presentation $P = \langle X|R \rangle$. By the uniformity of Higman's result (as described in the proof of [18, Theorem 12.18]) there is a uniform procedure that, for any i , outputs a finite set of words $S_i \subset X^*$ in 1-1 correspondence with X_i such that the subgroup $\langle S_i \rangle^{\overline{P}}$ is isomorphic to \overline{P}_i , and an explicit bijection $\phi_i : X_i \rightarrow S_i$ which extends to an isomorphism $\overline{\phi}_i : \overline{P}_i \rightarrow \langle S_i \rangle^{\overline{P}}$. That is, we can keep track of where each \overline{P}_i is sent in this embedding. So, given a recursive presentation $Q = \langle Y|S \rangle$ and a finite presentation $H = \langle Z|V \rangle$ such that $\overline{H} \cong \overline{Q}$, compute j such that $P_j = Q$ as recursive presentations (that is, number all Turing machines which read alphabet Y , and look for one identical to the description S). We know that $\overline{H} \cong \langle S_j \rangle^{\overline{P}}$. So use our algorithm to output a set map $\psi : Z \rightarrow S_j^*$ which extends to an isomorphism $\overline{\psi} : \overline{H} \rightarrow \langle S_j \rangle^{\overline{P}}$. But we can compose ψ with ϕ_j^{-1} to get the set map $\phi_j^{-1} \circ \psi : Z \rightarrow Y^*$ which extends to an isomorphism $\overline{\phi_j^{-1} \circ \psi} : \overline{H} \rightarrow \overline{Q}$. But this is impossible by theorem 4.2, so our assumed algorithm can't exist. \square

5. FURTHER WORK

The group G constructed in theorem 4.6 contains an embedded copy of every recursively presented group, so is definitely not coherent, let alone weakly effectively coherent. We ask the question of whether theorem 4.6 can be modified so that G is weakly effectively coherent, or even just coherent. Either of these would help make the result even more relevant, as we would like to resolve question 1 which was the motivation for this work.

REFERENCES

- [1] S. I. Adian, *Finitely presented groups and algorithms*, Dokl. Akad. Nauk SSSR **117**, 9–12 (1957).
- [2] G. Baumslag, D. Solitar, *Some two generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. **68**, 199–201 (1962).
- [3] W. W. Boone, *The word problem*, Ann. of Math., **70**, 207–265 (1959).
- [4] J. L. Britton, *The word problem for groups*, Proc. London Math. Soc. (3) **8**, 493–506 (1958).
- [5] M. Chiodo, *Finding non-trivial elements and splittings in groups*, J. Algebra. **331**, 271–284 (2011).
- [6] D. J. Collins *The word, power and order problems in finitely presented groups*, in Word Problems, eds. Boone, Cannonito, and Lyndon, Amsterdam, North-Holland, 401–420 (1973).
- [7] M. Dehn, *Über unendliche diskontinuierliche Gruppen* (German), Math. Ann. **69**, 116–144 (1911).
- [8] D. Groves, H. Wilton, *Enumerating limit groups*, Groups, Geometry and Dynamics **3**, 389–399 (2009).
- [9] D. Groves, H. Wilton, *Enumerating limit groups: A Corrigendum*, arXiv:1112.1223v1 (2011).
- [10] G. Higman, *Subgroups of finitely presented groups*, Proc. Royal Soc. London Ser. A **262**, 455–475 (1961).
- [11] R. Lyndon, P. Schupp, *Combinatorial Group Theory*, Springer, (2001).
- [12] W. Magnus, A. Karrass, D. Solitar, *Combinatorial group theory*, Dover, (2004).
- [13] C. F. Miller III, *On group theoretic decision problems and their classification*, Annals of Math. Study **68**, Princeton University Press, Princeton, NJ, 1971.
- [14] C.F. Miller III, *Decision problems for groups-survey and reflections*. Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., **23**, Springer, New York, 1–59 (1992).
- [15] P. S. Novikov, *On the algorithmic unsolvability of the word problem in group theory*, Trudy Mat. Inst. Steklov **44**, 1–143 (1955).
- [16] M. O. Rabin, *Recursive unsolvability of group theoretic problems*, Annals of Math. **67**, 172–194 (1958).
- [17] H. Rogers Jr, *Theory of recursive functions and effective computability*, MIT Press, (1987).
- [18] J. Rotman, *An introduction to the theory of groups*, Springer-Verlag, New York, (1995).

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